Maximize income and minimize inputs

If the joint production function is $x = q_1^2 + q_2^2$ and the selling prices are $p_1 = 20$ and $p_2 = 10$, the following is required:

1. Find the maximum income if the amount of inputs must be $x = 500$.
2. Find the minimum amount of input X that provides an income of 1500 monetary units.

Solutions

1. The income function is given by:

$$I = p_1 q_1 + p_2 q_2$$

The problem to solve is to maximize I subject to the input constraint: $500 = q_1^2 + q_2^2$

$$L = p_1 q_1 + p_2 q_2 + \lambda(500 - q_1^2 - q_2^2)$$

The first-order conditions:

$$L'_{q_1} = p_1 - \lambda 2 q_1 = 0$$
$$L'_{q_2} = p_2 - \lambda 2 q_2 = 0$$
$$L'_{\lambda} = 500 - q_1^2 - q_2^2 = 0$$

From the first two equations, we solve for λ and set them equal:

$$\frac{p_1}{2q_1} = \frac{p_2}{2q_2}$$

$$q_2 = \frac{p_2}{p_1} q_1$$

Insert into the third condition:

$$500 - q_1^2 - \left(\frac{p_2}{p_1} q_1\right)^2 = 0$$

$$500 = (1 + \frac{p_2^2}{p_1^2}) q_1^2$$

$$500 = \left(\frac{p_1^2 + p_2^2}{p_1^2}\right) q_1^2$$

$$q_1 = \sqrt{500} \frac{p_1}{\sqrt{p_1^2 + p_2^2}}$$

This is inserted into the expression for q_2:

$$q_2 = \frac{p_2}{p_1} \sqrt{500} \frac{p_1}{\sqrt{p_1^2 + p_2^2}}$$

$$q_2 = \sqrt{500} \frac{p_2}{\sqrt{p_1^2 + p_2^2}}$$

Replacing with $p_1 = 20$ and $p_2 = 10$:

$$q_1 = 20$$
$$q_2 = 10$$

This gives us an income of $I = 20 \times 20 + 10 \times 10 = 500$. Now we set up the bordered Hessian to verify that we are dealing with a maximum:

$$L''_{q_1 q_1} = -2\lambda$$
$$L''_{q_2 q_2} = -2\lambda$$
$$L''_{q_1 q_2} = L''_{q_2 q_1} = 0$$
$$g'_{q_1} = -2q_1$$
$$g'_{q_2} = -2q_2$$

We also obtain the value of λ with the first equation of the first-order conditions:

$$\lambda = \frac{p_1}{2q_1} = 0.5$$

$$|\bar{H}|= \begin{vmatrix} 0 & -2q_1 & -2q_2 \\ -2q_1 & -2\lambda & 0 \\ -2q_2 & 0 & -2\lambda \end{vmatrix} = \begin{vmatrix} 0 & -40 & -20 \\ -40 & -1 & 0 \\ -20 & 0 & -1 \end{vmatrix}$$

We calculate the determinant:

$$|H|= 40[(40)(-1)] - 20(-20) = 2000 > 0$$

This indicates that we are dealing with a maximum.

2. The minimization process is to set up the problem in reverse from the previous one:

$$L = q_1^2 + q_2^2 + \lambda(1500 - p_1q_1 - p_2q_2)$$

We calculate the first-order conditions:

$$L'_{q_1} = 2q_1 - \lambda p_1 = 0$$

$$L'_{q_2} = 2q_2 - \lambda p_2 = 0$$

$$L'_{\lambda} = 1500 - p_1q_1 - p_2q_2 = 0$$

From the first two equations, we get:

$$\frac{2q_1}{p_1} = \frac{2q_2}{p_2}$$

$$q_1 = \frac{p_1q_2}{p_2}$$

Insert into the third equation:

$$1500 - p_1\frac{p_1q_2}{p_2} - p_2q_2 = 0$$

$$1500 = q_2\left(\frac{p_1^2}{p_2} + p_2\right)$$

$$1500 = q_2\left(\frac{p_2^2 + p_1^2}{p_2}\right)$$

$$q_2 = \frac{p_2}{p_2^2 + p_1^2}1500$$

Insert this into the expression for q_1:

$$q_1 = \frac{p_1}{p_2}\frac{p_2}{p_2^2 + p_1^2}1500$$

$$q_1 = \frac{p_1}{p_2^2 + p_1^2}1500$$

Using the values $p_1 = 20$ and $p_2 = 10$:

$$q_1 = 30$$

$$q_2 = 60$$

This gives us a cost of:

$$30^2 + 60^2 = 4500$$

Now to check that we are dealing with a minimum, we solve the bordered Hessian, for which we find the following derivatives:

$$L''_{q_1q_1} = 2$$

$$L''_{q_2q_2} = 2$$

$$L''_{q_2q_1} = L''_{q_1q_2} = 0$$

$$g'_{q_1} = -p_1$$

$$g'_{q_2} = -p_2$$

$$|\bar{H}| = \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & 2 & 0 \\ -p_2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -20 & -10 \\ -20 & 2 & 0 \\ -10 & 0 & 2 \end{vmatrix}$$

We calculate the determinant:

$$|\bar{H}| = 20[-20 \times 2] - 10[20] = -1000 < 0$$

This verifies that we are dealing with a minimum.